Inclined convection in a porous Brinkman layer: linear instability and nonlinear stability

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Abstract

In this article we deal with thermal convection in an inclined porous layer when the fluid motion is modelled by the Brinkman law, inertial effects are also taken into account (i.e. the Vadasz number $Va$ is finite), and the physically significant rigid boundary conditions are imposed. This model is an extension of the work by Rees and Bassom [1], where the Darcy’s law is adopted, and only linear instability is investigated. It also completes the work of Falsaperla and Mulone [2], where the case of stress free boundary conditions is studied and the inertial terms are absent. In this model the layer is heated from below. In the resulting basic laminar solution the temperature is linear with respect to the transverse coordinate, and the velocity is a combination of hyperbolic and polynomial functions, which makes the linear and nonlinear analysis much more complex.

The original features of the paper are the following: i) we study linear instability and nonlinear stability with respect to three-dimensional perturbations; ii) we study nonlinear stability with the Lyapunov second method and for the first time in the literature (in the case of inclined layers) we compute the critical nonlinear Rayleigh regions by solving the associated variational maximum problem; iii) we give some estimates of global nonlinear asymptotical stability; iv) for longitudinal perturbations we prove the coincidence of linear and nonlinear critical Rayleigh numbers; v) we provide many three-dimensional critical surfaces for linear and non linear stability; vi) we study linear instability and nonlinear stability also with the presence of the inertial terms $Va < +\infty$; vii) we prove the stabilizing effects of the Darcy’s number, the inclination angle, and the inertial term $G$ up to an inclination angle of about 65°.
1 Introduction

Thermal convection in a horizontal fluid or in a porous Darcy’s layer has been studied by many authors because of its applications to geophysics, engineering and many other areas (see the monographs by Straughan [3,4], Nield and Bejan [5], and the references therein).

Also inclination of the layer plays an important role in convection of fluids or of fluid-saturated porous layers (environmental situations, geothermal systems, filtration processes, ground water pollution, solidification of castings), and it has been considered by many authors e.g. Chen and Pearlstein [6], Ostrach [7], Takashima [8], Woods and Linz [9], Vasseur et al. [10], Sun [11], Ortiz-Preza and Dvalos-Orozcob [12], Subramanian et al. [13], Falsaperla et al. [14–16], Alsabery et al. [17], Bories and Combarnous [18], Weber [19], Vasseur et al. [20], Rees [21], Rees et al. [22], Nield [23], Barletta and Storelletten [24], Barletta and Rees [25], Pathak and Singh [26], Barletta and Celli [27], Barletta [28], Barletta and Rees [29], Barletta and Celli [30], Rees and Bassom [31] and Wen and Chini [32].

As observed by some authors, the dynamics of a fluid in porous media cannot be adequately modelled by the simple Darcy’s law. Vasseur et al. [20] study analytical solutions in an inclined two-dimensional cavity, confined on all sides by an impermeable rectangular box. They note: “however, it has been observed that boundary effects, not included in the Darcy’s law model, may become significant for fluid flows in saturated porous media with high permeabilities”. Rees [21] observes: “in the last 10 years there has been a great upsurge of interest in determining the effects of extensions to Darcy’s law since practical applications involve media for which Darcy’s law is inadequate”, see also Nield and Bejan [5, Sect. 1.5.3, Sect. 6.6], Straughan [33], Yadav et al. [34, 35]. Also inertial effects play an important role (see [5, Sect. 1.5.1] and [36]).

An area of particular interest for thermal convection in an inclined plane is that of landslides, Montrasio et al. [37] or more generally land deformation involving thermal gradients, Sanavia and Schrefler [38], Hammond and Barr [39]. Also bidisperse porous media are crucial to understanding the physics of underground drinking water supplies, see Zuber and Motyka [40], Ghasemizadeh et al. [41] and Kim and Moridis [42]. Bidispersive inclined and vertical convection has been studied in Nield and Kuznetsov [43], Falsaperla et al. [44], Gentile and Straughan [45]. There are also some new progress on nanofluid convection in porous medium [46–50].

The aim of this paper is to study the linear instability and the nonlinear stability of a fluid saturating an inclined porous layer heated from below by assuming that the fluid motion obeys the Darcy-Brinkman law and the
boundary conditions are of rigid type. The paper is an extension of the work by Rees and Bassom [1], where the Darcy’s law is adopted, it is also an extension of the work of Falsaperla and Mulone [2], where the case of stress free boundary conditions is studied. We also add the inertial terms to the Darcy-Brinkman equation that in some cases cannot be neglected (see Nield and Bejan [5]), and we consider three-dimensional perturbations.

In the Darcy-Brinkman model the basic motion is a combination of hyperbolic and polynomial functions. We investigate numerically, with the Chebyshev collocation method, the linear instability of such basic motion for transverse perturbations, for longitudinal perturbations and for general three-dimensional perturbations. We also study, with the Lyapunov second method, the nonlinear stability by giving estimates of nonlinear stability thresholds. Moreover, by introducing an energy Lyapunov functional, we give nonlinear stability results and, as far as we know for the first time in the literature for this case, we compute the nonlinear stability thresholds by solving the Euler-Lagrange equations associated to the maximum problem. In particular, for longitudinal perturbations, by defining another appropriate Lyapunov functional, we prove the coincidence of linear and nonlinear critical Rayleigh numbers for these perturbations.

In this paper we consider only heating from below. The case of heating from above can be done in a similar way as for the stress free boundary conditions, see [2].

The plan of the paper is the following: in Section 2 we introduce the basic equations for thermal convection in a porous medium with Brinkman law. In Section 3 we derive the basic solution for thermal convection in an inclined layer and we write the perturbation equations. Section 4 deals with the linear instability in detail. In particular, we obtain the critical linear instability numbers by the Chebyshev collocation method. For longitudinal perturbations we give analytical results which coincide with the numerical ones. In Section 5 we give some nonlinear stability results for general three dimensional perturbations and for longitudinal perturbations. The final Section is reserved to comments of the numerical results and conclusions.

2 Basic equations

Let us consider an inclined layer of porous medium inclined of an angle $\varphi$ with respect to the horizontal plane (cf. the geometrical configuration of Rees and Bassom [1]), let $\epsilon$ be the porosity of the medium, and let $x$ be measured in the longitudinal direction along the layer, while $z$ is orthogonal to the layer, which has boundaries at $z = -d/2$ and $z = d/2$. Denote by
the seepage velocity of the fluid in the medium and also assume that the system has a non-homogeneous temperature $T$. The equations governing the velocity field including inertial terms are given by (see e.g. [5, pag. 9, eq. (1.9)])

$$
\frac{\rho_0}{\epsilon} \frac{\partial U_i}{\partial t} = -\frac{\mu}{K} U_i + \bar{\mu} \Delta U_i - p, i + \rho g_i
$$  \hspace{1cm} (1)

$$
U_{i,i} = 0,
$$  \hspace{1cm} (2)

where $i = 1, 2, 3$, and $g_i = -g \lambda_i$ with $\lambda = (\sin \varphi, 0, \cos \varphi)^T$.

Throughout this work we employ standard notation: the commas denote partial derivation with respect to the corresponding variables $x_i$ while $\Delta$ is the Laplace operator.

In these equations $\mu$ is the fluid viscosity, $K$ is the permeability, $p$ is the pressure, $\alpha$ is the thermal expansion coefficient, $T_0$ is a reference temperature, $\rho_0$ is the reference density at temperature $T_0$, and

$$
\rho = \rho_0 [1 - \alpha (T - T_0)].
$$  \hspace{1cm} (3)

The second term of the r.h.s. of equation (1) is the Brinkman term, and $\bar{\mu}$ denotes an effective viscosity.

The behaviour of the temperature field is described in Nield and Bejan [5], and in Straughan [33] and it evolves in time according to equation

$$
\frac{1}{M} \frac{\partial T}{\partial t} + U_i T_{,i} = \kappa \Delta T,
$$  \hspace{1cm} (4)

where $M$ is the inverse of the heat capacity ratio, see Nield and Bejan [5, p. 243, (6.14b)] and $\kappa$ is the thermal diffusivity of the fluid phase, see [5] or [2, Sec. 2].

## 3 Basic solution and perturbation equations

The equations (1), (2), (4) can be summarized in the system

$$
\begin{aligned}
\frac{\rho_0}{\epsilon} \frac{\partial U_i}{\partial t} &= -\frac{\mu}{K} U_i + \bar{\mu} \Delta U_i - p, i + \rho_0 \alpha (T - T_0) \lambda_i - \rho_0 g \lambda_i \\
\frac{1}{M} \frac{\partial T}{\partial t} + U_i T_{,i} &= \kappa \Delta T \\
U_{i,i} &= 0.
\end{aligned}
$$  \hspace{1cm} (5)

We assume that the velocity field is subject to rigid boundary conditions, that is the values of $U$ are given and constant when $z = \pm d/2$. For the
temperature, we assume the boundary conditions
\[ T(-d/2) = T_L, \quad T(d/2) = T_U \]
with \( T_L, T_U \) positive constants and \( T_L \geq T_U \) since we consider the case of heating from below.

To rewrite the system in a non dimensional form, we introduce the temperature gradient \( \beta = (T_L - T_U)/d \), and we assume the reference temperature \( T_0 \) to be \( T_0 = (T_L + T_U)/2 \). Introducing the new units for length, time, velocity, temperature and pressure respectively
\[
x = x^* d; \quad t = t^* \frac{d^2}{MK}; \quad U = U^* \frac{\kappa}{d}; \quad T = T_0 + \beta d T^*; \quad p = \frac{p^* \mu K}{K}
\]
and the Darcy, the Vadasz and the Rayleigh numbers
\[
\text{Da} = \frac{\bar{\alpha}}{\mu d^2}, \quad \text{Va} = \frac{\epsilon \nu d^2}{k K}, \quad R = \frac{\rho_0 \alpha \beta g d^2 K}{\mu K}.
\]
Denoting by \( \bar{\alpha} = \frac{d \rho_0 g K}{\mu K} \) and dropping the “stars” on the new units, system (5) can be rewritten as
\[
\begin{cases}
M \frac{\partial U_i}{\partial t} = -p, - U_i + \text{Da} \Delta U_i + (RT - \bar{\alpha}) \lambda_i, \\
\frac{\partial T}{\partial t} + U_i T_i = \Delta T, \\
U_{i,i} = 0.
\end{cases}
\]
(6)

Note that in the new coordinates the \( z \)-interval is \([-1/2, 1/2]\) and the boundary conditions on the temperature are
\[ T(-1/2) = 1/2, \quad T(1/2) = -1/2. \]

The stationary laminar solutions to the systems of equations are functions of the form \( \mathbf{U} = (\bar{U}(z), 0, 0) \), and \( \bar{T}(z) \) linear in \( z \) and \( \bar{p} \) given below. The stationary laminar solution of (6) is
\[ \bar{T} = -z. \]
(7)

By substituting \( \bar{T} \) in (6)1, we obtain (see [2])
\[ \bar{p}(x, y, z) = p(x, z) = p(x) - \frac{1}{2} R \cos \varphi z^2 - \bar{\alpha} \cos \varphi z \]
(8)
with
\[ p(x) = (\sigma - \dot{\alpha} \sin \varphi) x + p_0. \]
we observe that \( \sigma \) is a real number that represents a pressure gradient along \( x \), while \( p_0 \) is a constant.

To satisfy (6)_1, the velocity field of a stationary laminar solution is hence
\[ \bar{U}(z) = c_1 \cosh(\gamma z) + c_2 \sinh(\gamma z) - R \sin \varphi z - \sigma, \]
where the constants \( c_1, c_2 \) depend on the boundary conditions on \( U \) and we have introduced the symbol \( \gamma = Da^{-1/2} \).

Since in this work we consider here rigid conditions
\[ U(-1/2) = \bar{u}_1, \quad U(1/2) = \bar{u}_2. \]
the velocity field of a stationary laminar solution can be rewritten in the form
\[ \bar{U}(z) = \frac{\bar{u}_1 + \bar{u}_2 + 2 \sigma \cosh(\gamma z)}{2 \cosh(\gamma/2)} + \frac{\bar{u}_2 - \bar{u}_1 + R \sin \varphi \sinh(\gamma z)}{2 \sinh(\gamma/2)} - R z \sin \varphi - \sigma. \]
As customary, we call \((\bar{U}, \bar{p}, \bar{T})\) basic solution.

The form of the basic kinetic field \( \bar{U}(z) \) above is the more general for rigid boundary conditions, but in the numerical part of the article we will focus on the physical case of fixed boundaries, i.e. \( \bar{u}_1 = \bar{u}_2 = 0 \). For simplicity, in this work, the numerics is also performed posing \( \sigma = 0 \). The interesting physical case \( \sigma \neq 0 \) (“Poiseuille-like” motion) will be studied in a future work.

We introduce perturbations to the basic solution and write
\[ U_i = \bar{U}_i + u_i, \quad T = \bar{T} + \theta, \quad p = \bar{p} + \pi, \]
and assume that \( u_i, \theta \) and \( \pi \) are periodic functions in \( x \) and \( y \) of period \( 2\pi/a \) and \( 2\pi/b \) respectively. We denote by
\[ \Omega = \left[ 0, \frac{2\pi}{a} \right] \times \left[ 0, \frac{2\pi}{b} \right] \times \left[ -\frac{1}{2}, \frac{1}{2} \right] \]
the periodicity cell. Denoting \( u_1 = u, \ u_2 = v, \ u_3 = w, \) the perturbation equations are \( \nabla \cdot \mathbf{u} = 0 \) and
\[
\begin{cases}
G \frac{\partial u_i}{\partial t} = -u_i - \pi_i + Da \Delta u_i + R \theta \lambda_i \\
\frac{\partial \theta}{\partial t} + \bar{U}(z) \theta_x - w + u_i \theta_i = \Delta \theta
\end{cases}
\]
where \( G = M/Va \) is the inertial coefficient. The boundary conditions are
\[ u = 0, \quad v = 0, \quad w = 0, \quad \theta = 0, \]
for \( z = \pm 1/2 \).
4 Linear instability

To investigate the linear instability we consider the linearization of equations (10) that are

\[
\begin{align*}
G \frac{\partial u_i}{\partial t} &= -u_i - \pi_{,i} + Da \Delta u_i + R \theta \lambda_i \\
\frac{\partial \theta}{\partial t} + \bar{U}(z) \theta_x - w - \Delta \theta &= 0 \\
u_{i,i} &= 0.
\end{align*}
\] (11)

To remove the pressure $\pi$ we consider the double curl of the first equation so that the third component of the resulting equation is

\[-G \Delta w_t = \Delta w - Da \Delta^2 w + R \sin \varphi \theta_{,xx} - R \cos \varphi (\theta_{,xx} + \theta_{,yy}).\]

We introduce normal modes writing the perturbations as $f(x,y,z,t) = f(z)e^{i(ax+by)+ct}$ where $f$ is $w$ or $\theta$, $a$ and $b$ are the wave numbers, $\alpha^2 = a^2 + b^2$, and $c$ is the time decay coefficient. Eq. (11) becomes

\[
\begin{align*}
-Gc(D^2 - \alpha^2)w &= (D^2 - \alpha^2)w - Da(D^2 - \alpha^2)^2w + \\
+iaR \sin \varphi D\theta + R \cos \varphi \alpha^2 \theta \\
c \theta - w + ia\bar{U}(z) \theta - (D^2 - \alpha^2)\theta &= 0,
\end{align*}
\] (12)

where $D$ denotes the derivative with respect to $z$.

In what follows we will solve equations (12) numerically by means of a Chebyshev collocation method, using the boundary conditions

\[w = 0, \quad w' = 0, \quad \theta = 0, \quad \text{when } z = \pm 1/2.\]

4.1 Transverse and longitudinal perturbations

Let us begin considering spanwise perturbations (also called transverse perturbations), i.e. independent on $y$. Such perturbations have a rich dynamics (see [1]) and they are sometimes the most destabilizing perturbations. As it is well known, spanwise perturbations are very important also in laminar flows such as the Couette and Poiseuille flows (see Orr [51], Joseph [52], Falsaperla et al. [54]). The spanwise perturbations can be investigated setting $b = 0$ in equations (12).

In Figure 1 we plot the Rayleigh numbers as functions of the wave number $a$ for Darcy’s number $Da = 0.01$ (for other Darcy’s numbers the results are similar), $G = 0$, and different inclination angles. In Figure 9 we consider the case $G \neq 0$. 

7
Assuming that the perturbations are streamwise (also called longitudinal perturbations), i.e. independent of $x$, the divergence-free condition on the velocity field is $v_y + w_z = 0$. Observe that the hypothesis does not mean that $u = 0$ (in fact, this cannot be true unless $\varphi = 0$ or the perturbation reduces to zero). Nevertheless, in this case, it can be proved that the time evolution of $u$ can be controlled by the time evolution of $\theta$. From (12) it is hence sufficient to study the linear system:

\[
\begin{aligned}
-Gc(D^2 - b^2)w &= (D^2 - b^2)w - Da(D^2 - b^2)^2 w + R \cos \varphi b^2 \theta, \\
c \theta - w - (D^2 - b^2)\theta &= 0.
\end{aligned}
\] (13)

An easy calculation shows that the decay $c$ is a real number (it suffices to multiply (13)$_1$ by the complex conjugate of $w$ and integrate over $\Omega$, and to multiply (13)$_2$ by $R \cos \varphi b^2$ times the complex conjugate of $\theta$ and integrate over $\Omega$, and add the equations).

Setting $c = 0$, i.e. assuming to be at the criticality, equations (13) become the same as those for a horizontal layer ($\varphi = 0$) where now we have $R \cos \varphi$ instead of $R$. Numerical results with Chebyshev collocation method give the critical Rayleigh number $R_B$. Hence the critical parameter, with respect to longitudinal perturbations, is given by

\[
R_{Long} = \frac{R_{B}}{\cos \varphi},
\] (14)

where $R_B$ is the critical Rayleigh number for linear instability in the case of a horizontal layer ($\varphi = 0$) and rigid boundary conditions.

We shall show in Sec. 5 that these linear results coincide with the nonlinear ones because the linear equations (13), with $c = 0$, are the same as the Euler-Lagrange equations for the corresponding maximum problem with $m = 1$ (see formula (40) below).

In Figures 2a, 2b and 2c we plot the critical linear Rayleigh numbers associated to longitudinal and transverse perturbations for different values of Darcy’s numbers.

### 4.2 3-D perturbations

We also study the system (12) with both $a$ and $b$ not zero. In fact, in our case (Brinkman law) the validity of a Squire theorem has not been proved. In Figures 5 and 6 we plot the surfaces of the linear Rayleigh numbers as function of the wave numbers $a$ and $b$. 

8
5 Nonlinear stability

In this section we give sufficient conditions for global nonlinear asymptotic stability.

We first give some estimates of the nonlinear critical Rayleigh number, then we study the maximum problem arising from an energy equation. This problem is very difficult in general because the inclination of the layer introduces partial derivatives with respect to \( x \) and we must solve the Euler-Lagrange equations very carefully (in a similar way of the celebrated Orr equations for parallel flows in hydrodynamics (see Orr [51], Drazin and Reid [55], pp. 162-163, Falsaperla et. al [54])). As far as we know this is an open problem that is studied here for the first time.

We also give sufficient nonlinear stability conditions in the case of longitudinal perturbations, i.e. perturbations which do not depend on \( x \). In this case we prove the coincidence of linear and nonlinear (with a suitable Lyapunov functional) critical thresholds \( R_{\text{Long}} \) with respect to this class of perturbations.

5.1 Estimates for critical nonlinear Rayleigh numbers

i) Darcy-Brinkman flows without inertial terms (the case \( G = 0 \))

We start with equations (10), so that we have the nonlinear perturbations system

\[
\begin{align*}
-u_i - \pi_{,i} + Da \Delta u_i + R \theta \lambda_i &= 0, \\
\theta_{,t} + U(z) \theta_{,x} - w + u_i \theta_{,i} &= \Delta \theta.
\end{align*}
\]  

(15)

We suppose that we have rigid boundaries with assigned temperatures:

\[
w = 0, \quad w' = 0, \quad \theta = 0, \quad \text{on} \quad z = \pm \frac{1}{2},
\]

and \( u, \theta \) and \( \pi \) satisfy a plane tiling periodicity in \( x \) and \( y \), such that the solution has a periodicity cell \( \Omega \).

Proceeding as in [2], multiplying (15) by \( u \) and integrating over \( \Omega \), by using Cauchy–Schwarz inequality and the arithmetic and geometric mean inequality, we have

\[
\|u\|^2 = R \cos \varphi(w, \theta) + R \sin \varphi(u, \theta) - Da \|\nabla u\|^2 \leq \frac{\epsilon}{2} \|w\|^2 + \frac{1}{2\epsilon} R^2 \cos^2 \varphi \|\theta\|^2 + \frac{\epsilon_1}{2} \|u\|^2 + \frac{1}{2\epsilon_1} R^2 \sin^2 \varphi \|\theta\|^2 - Da \|\nabla u\|^2.
\]  

(16)

In (16) \((\cdot, \cdot)\) and \(\|\cdot\|\) are the inner product and norm in \( L^2(\Omega) \).
By using the Poincaré inequality, we have
\[
\|u\|^2 = R \cos \varphi (w, \theta) + R \sin \varphi (u, \theta) - Da \|\nabla u\|^2 \leq \frac{\epsilon}{2} \|w\|^2 + \frac{1}{2\epsilon} R^2 \cos^2 \varphi \|\theta\|^2 + \frac{1}{2\epsilon_1} \|u\|^2 + \frac{1}{2\epsilon_1} R^2 \sin^2 \varphi \|\theta\|^2 - Da \pi^2 \|u\|^2.
\] (17)

From this it follows
\[
B \|u\|^2 - \frac{\epsilon_1}{2} \|u\|^2 - \frac{\epsilon}{2} \|v\|^2 \leq \frac{R^2}{2} \|\theta\|^2 \left( \frac{\cos^2 \varphi}{\epsilon} + \frac{\sin^2 \varphi}{\epsilon_1} \right)
\] (18)

with \( B = 1 + Da \pi^2 \). By choosing \( \epsilon_1 = 2B \) and \( \epsilon < 2B \), we obtain
\[
(2B - \epsilon) \|w\|^2 \leq R^2 \left( \frac{1}{\epsilon} \cos^2 \varphi + \frac{1}{2B} \sin^2 \varphi \right) \|\theta\|^2.
\] (19)

Now, multiply (15) by \( \theta \) and integrate over \( \Omega \). After some integrations by parts and use of the boundary conditions we obtain the energy equation
\[
\dot{V} = (w, \theta) - \|\nabla \theta\|^2,
\] (20)

where the energy is given by
\[
V = \frac{1}{2} \|\theta\|^2.
\]

It is worth observing that the term involving \( \theta_{,x} \) integrates to zero as do the cubic nonlinear terms which arise.

From (20) we obtain the estimate:
\[
\dot{V} \leq \|w\| \|\theta\| - \|\nabla \theta\|^2 \leq \|w\| \|\theta\| - \pi^2 \|\theta\|^2
\]
\[
\leq \left\{ \frac{R}{\sqrt{2B - \epsilon}} \left[ \frac{\cos^2 \varphi}{\epsilon} + \frac{\sin^2 \varphi}{2B} \right]^{1/2} - \pi^2 \right\} \|\theta\|^2,
\] (21)

and the condition
\[
R^2 < \max_{\epsilon \in (0, 2B)} \frac{\pi^4 (2B - \epsilon)(1 + Da \pi^2)}{\cos^2 \varphi + \frac{\sin^2 \varphi}{2B}},
\]
i.e.,
\[
R < \frac{2\pi^2 (1 + Da \pi^2)}{\cos \varphi + 1},
\] (22)

implies global nonlinear asymptotic stability of the basic motion. From this result it appears a stabilizing effect (from the nonlinear point of view) both from the Darcy’s number and the inclination of the layer.
ii) Darcy-Brinkman flows with inertial term (the case $G \neq 0$)

A similar estimate can be obtained in this case. In fact defining the Lyapunov functional

$$V = \frac{G}{2} \|u\|^2 + \frac{1}{2} R \bar{\alpha} \|\theta\|^2,$$

where $\bar{\alpha}$ is a positive parameter to be chosen, we have

$$\dot{V} \leq \left( \frac{R^2 (\cos \varphi + \bar{\alpha})^2}{2 \epsilon} + \frac{R^2 \sin^2 \varphi}{2 \epsilon_1} - R \bar{\alpha} \pi^2 \right) \|\theta\|^2 + \left( \frac{\epsilon}{2} - B \right) \|w\|^2 + \left( \frac{\epsilon_1}{2} - B \right) \|u\|^2. \quad (23)$$

Choosing $\epsilon = \epsilon_1 = 2B - \sigma_1$ (with $\sigma_1 > 0$ arbitrary) and $\bar{\alpha} = 1$, we easily obtain that the condition

$$R < \frac{2 \pi^2 (1 + Da \pi^2)}{1 + \cos \varphi}$$

implies global nonlinear stability.

We note that this estimate gives, for a horizontal layer and $Da = 0$, the stability condition $R < \frac{\pi^2}{2}$ (instead the critical instability Rayleigh number is $4 \pi^2$), see also Fig. 7.

### 5.2 Critical energy Rayleigh numbers obtained from a maximum problem

We introduce the Lyapunov functional

$$V_G = \frac{G}{2} \|u\|^2 + \frac{1}{2} R \|\theta\|^2, \quad (24)$$

and study the variational problem to obtain the best value of the critical nonlinear Rayleigh number $R_{V_G}$.

Let $\mathcal{S}$ be the space of the *kinematically admissible fields*

$$\mathcal{S} = \{ u, v, w, \theta \in W^{1,2}(\Omega), \ u = v = w = \theta = 0 \text{ when } z = \pm 1/2, \text{ periodic in } x \text{ and } y, \ u_x + v_y + w_z = 0, \|u\|^2 + \|\theta\|^2 > 0 \}, \quad (25)$$

define

$$m = \max_\mathcal{S} \frac{R (\cos \varphi + 1)(w, \theta) + R \sin \varphi (u, \theta)}{\|u\|^2 + Da \|\nabla u\|^2 + R \|\nabla \theta\|^2}, \quad (26)$$
from (24) and (26), we obtain

\[ \dot{V}_G = I - D \leq (m - 1)D, \]  

(27)

where

\[ I = R(\cos \varphi + 1)(w, \theta) + R\sin \varphi(u, \theta) \]

and

\[ D = ||u||^2 + Da||\nabla u||^2 + R||\nabla \theta||^2. \]

The Euler-Lagrange equations of the maximum problem (26) are

\[
\begin{align*}
R(\cos \varphi + 1) \theta k + R\sin \varphi \theta i - 2m(u - Da\Delta u) &= \nabla \bar{p} \\
R(\cos \varphi + 1) w + \sin \varphi u + 2m \Delta \theta &= 0,
\end{align*}
\]

(28)

where \( \bar{p} \) is a Lagrange multiplier and \( i \) and \( k \) are unit vectors in the directions \( x \) and \( z \), respectively. The nonlinear stability condition are be

\( m < 1 \)

and criticality is reached when \( R = R_{V_G} \) in (28) with \( m = 1 \).

We can remove the multiplier \( \bar{p} \) by taking the third component of the curl and the double curl of the first equation. We hence have

\[
\begin{align*}
-R(\cos \varphi + 1) \Delta \theta - 2(\Delta w - Da\Delta w) &= 0, \\
-R \sin \varphi \theta_{xy} - 2(\zeta - Da\Delta \zeta) &= 0, \\
(cos \varphi + 1) w + \sin \varphi u + 2\Delta \theta &= 0.
\end{align*}
\]

(29)

Now recall the relations between the components of the velocity field (in the case of plan-form, see Chandrasekhar [56, pag. 24]):

\[
\begin{align*}
\zeta &= v_x - u_y, \\
u &= \frac{w_{xz} + \zeta_{yx}}{\alpha^2}, \\
v &= \frac{w_{yz} - \zeta_{xy}}{\alpha^2}.
\end{align*}
\]

(30)

By taking the partial derivative of the second equation of (29), and defining \( h = \zeta_{xy} \) we have

\[
\begin{align*}
-R(\cos \varphi + 1) \Delta \theta + R \sin \varphi \theta_{xx} + 2(\Delta w - Da\Delta w) &= 0, \\
-R \sin \varphi \theta_{yy} - 2(h - Da\Delta h) &= 0, \\
(cos \varphi + 1) w + \frac{\sin \varphi}{\alpha^2} (w_{xz} + h) + 2\Delta \theta &= 0.
\end{align*}
\]

(31)

12
As before write $f(x, y, z) = f(z)e^{i(ax+by)}$ where $f$ is $h$, $w$ or $\theta$

\[
\begin{aligned}
R(\cos \varphi + 1)\alpha^2 \theta + iaR \sin \varphi D\theta + 2[(D^2 - \alpha^2)w - Da(D^2 - \alpha^2)^2w] = 0 \\
R \sin \varphi b^2 \theta - 2[h - Da(D^2 - \alpha^2)h] = 0 \\
(cos \varphi + 1)w + \frac{\sin \varphi}{\alpha^2}(ia Dw + h) + 2(D^2 - \alpha^2)\theta = 0.
\end{aligned}
\]

(32)

We note that the same system (32) is obtained also in the case $G = 0$.

To find the solutions of this system with the appropriate boundary conditions is a very difficult problem to handle in general (with both $a$ and $b$ different from 0). However, we have solved it with the Chebyshev collocation method. The results we obtain are represented in Figures 7 and 8 and they are commented in Section 6.

In particular, from these results, we see that the linear and nonlinear critical values are very close and they are obtained for streamwise or spanwise perturbations. However, we shall show below that, for streamwise perturbations, we can choose another Lyapunov functional, equivalent to $V_G$, such that the linear and nonlinear values with respect to these perturbations coincide.

5.3 Longitudinal perturbations

The system of equations when the perturbations are longitudinal are the following:

\[
\begin{aligned}
G\frac{\partial u}{\partial t} &= -u + Da \Delta u + R \sin \varphi \theta \\
G\frac{\partial v}{\partial t} &= -v + Da \Delta v - \frac{\partial p}{\partial y} \\
G\frac{\partial w}{\partial t} &= -w + Da \Delta w - \frac{\partial p}{\partial z} + R \cos \varphi \theta \\
\theta_x + \bar{U}(z) \theta_x - w + u_i \theta_i &= \Delta \theta.
\end{aligned}
\]

(33)

We observe that for these equations the time evolution of $\|u\|$ can be controlled by $\|\theta\|$, e.g. for $G = 0$, taking the scalar product of (33\textsubscript{1}) with $u$, we have

$$
\|u\|^2 + Da\|\nabla u\|^2 = R \sin \varphi(u, \theta) \leq R \sin \varphi\|u\|\|\theta\|.
$$

By using Cauchy inequality we have

$$
\|u\|^2 \left[1 - \frac{R^2 \sin^2 \varphi}{2\epsilon}\right] \leq \frac{\epsilon}{2} \|\theta\|^2.
$$
Choosing $\epsilon = R^2 \sin^2 \varphi$ we have

$$\|u\|^2 \leq R^2 \sin^2 \varphi \|\theta\|^2.$$  

A similar result can be obtained for $G \neq 0$. Thus we can use as Lyapunov functional (for $v$, $w$ and $\theta$)

$$V_1 = \frac{G(\|v\|^2 + \|w\|^2) + R \cos \varphi \|\theta\|^2}{2} \quad (34)$$

In this case, from (34), we obtain

$$\dot{V}_1 = -R \cos \varphi \|\nabla \theta\|^2 + 2R \cos \varphi (w, \theta)$$

$$- (\|v\|^2 + \|w\|^2) - Da(\|\nabla v\|^2 + \|\nabla w\|^2) \quad (35)$$

Let $S_1$ be the space of the kinematically admissible fields

$$S_1 = \{v, w, \theta \in W^{1,2}(V), v = w = \theta = 0 \text{ on the boundaries,} \quad \text{periodic in} \ x \ \text{and} \ y, \ v_y + w_z = 0, \|v\| + \|w\| + \|\theta\| > 0\} \quad (36)$$

Defining

$$m = \max_{S_1} \frac{2R \cos \varphi (w, \theta)}{R \cos \varphi \|\nabla \theta\|^2 + \|v\|^2 + \|w\|^2 + Da(\|\nabla v\|^2 + \|\nabla w\|^2)} \quad (37)$$

from (35) we obtain

$$\frac{d}{dt} \left[ G(\|v\|^2 + \|w\|^2) + \frac{R \cos \varphi \|\theta\|^2}{2} \right] \leq$$

$$\leq (m - 1)[R \cos \varphi \|\nabla \theta\|^2 + \|v\|^2 + \|w\|^2 + Da(\|\nabla v\|^2 + \|\nabla w\|^2)] \quad (38)$$

The Euler-Lagrange equations of the maximum problem (37) are

$$\begin{cases} R \cos \varphi \theta_k - m(u - Da \Delta u) = \nabla \tilde{p} \\ w + m \Delta \theta = 0 \end{cases} \quad (39)$$

where $\tilde{p}$ is a Lagrange multiplier. We remove the multiplier $\tilde{p}$ by taking $\text{curlcurl}$ of the first equation and the third component of the resulting equation. We have

$$\begin{cases} -R \cos \varphi \theta_{yy} + m(\Delta w - Da \Delta \Delta w) = 0 \\ w + m \Delta \theta = 0 \end{cases} \quad (40)$$
Table 1: Linear critical longitudinal - L - and transverse - T - Rayleigh numbers for different inclination angles and Darcy’s numbers and rigid boundary conditions.

This system coincides with the linear system (13) where we put $c = 0$ and with the Euler-Lagrange system for the horizontal layer with $R$ substituted by $R \cos \phi$.

The stability condition $m < 1$ is equivalent to

$$R < \frac{R_B}{\cos \phi},$$

where $R_B$ is the critical Rayleigh number for the horizontal layer.

Assume that condition (41) is satisfied and the perturbations to an inclined layer heated from below are longitudinal. Then the basic motion is nonlinearly asymptotically exponentially stable with respect to longitudinal perturbations.

In Tables 1 and 2, we give some critical Rayleigh numbers (linear and nonlinear with respect to the Lyapunov functional (23) for different Darcy’s numbers and inclination angles. For $\phi = 0$ we obtain the same results as in Rees [21], Fig 1.

6 Numerical results and conclusions

In this section we present the results for the numerical solution of linear and nonlinear equations for transverse, longitudinal and three-dimensional perturbations. We employ the Chebyshev collocation method and allow for

<table>
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<tr>
<th>Inclination</th>
<th>$\phi = 0^\circ$ - L</th>
<th>$\phi = 0^\circ$ - T</th>
<th>$\phi = 5^\circ$ - L</th>
<th>$\phi = 5^\circ$ - T</th>
<th>$\phi = 30^\circ$ - L</th>
<th>$\phi = 30^\circ$ - T</th>
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<td>$Da = 0$</td>
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<td>39.478</td>
<td>39.629</td>
<td>39.859</td>
<td>45.586</td>
<td>72.572</td>
<td>78.597</td>
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<td>60.398</td>
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<td>215.06</td>
<td>215.882</td>
<td>218.133</td>
<td>248.33</td>
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<tr>
<td>$Da = 1$</td>
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<td>1752.21</td>
<td>1758.904</td>
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<td>13840.041</td>
<td>3504.421</td>
<td>34697.504</td>
<td>20104.360</td>
<td>418109.272</td>
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</table>
Inclination \( \varphi = 0^\circ \) - L
\( \varphi = 0^\circ \) - T
\( \varphi = 5^\circ \) - L
\( \varphi = 5^\circ \) - T
\( \varphi = 30^\circ \) - L
\( \varphi = 30^\circ \) - T
\( \varphi = 60^\circ \) - L
\( \varphi = 60^\circ \) - T
\( \varphi = 85^\circ \) - L
\( \varphi = 85^\circ \) - T

<table>
<thead>
<tr>
<th>Inclination</th>
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<th>Da = 0.01</th>
<th>Da = 0.1</th>
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Table 2: Nonlinear critical energy stability with Lyapunov functional (24): longitudinal - L - and transverse - T - Rayleigh numbers for different inclination angles and Darcy’s numbers and rigid boundary conditions.

minimization in both \( a \) and \( b \) to find the minimum critical values of the Rayleigh numbers in the linear and nonlinear cases.

In Figure 1 we plot the critical Rayleigh number as function of the inclination angle \( \varphi \) for different Darcy’s numbers for transverse (solid lines) and longitudinal (dotted lines) perturbations. We also assume that the inertial term \( G = 0 \).

We have to remark that for \( Da = 0 \) (Darcy’s law) we obtain the same results as Rees and Bassom [1]. In particular, for inclination angles greater than \( \varphi^* = 31,469^\circ \) (calculation made with 30 Chebyshev polynomials) the transverse perturbations are always stable. For Darcy’s numbers greater than zero, such phenomenon does not appear. Instead, for given Darcy’s numbers lower than \( \varphi^* \) (depending on \( Da \)) there is a jump-discontinuity in the critical Rayleigh numbers for transverse perturbations. Moreover we note that the critical Rayleigh numbers for longitudinal perturbations are below those for transverse perturbations for any Darcy’s number.

In Figures 2a - 2c, we plot the critical curves in the plain \( (R,a) \) for different inclination angles \( \varphi \) and a fixed Darcy’s number \( Da = 0.01 \). The range of angles has been chosen to show the origin of the discontinuity of the critical Rayleigh number \( R_c \) observed in Fig. 1. We note first that the critical curves for angles slightly less then \( 25.5^\circ \) are not connected, and the minimum of \( R \) is located on a ”stability island”, which disappears for larger angles. We note also that after this discontinuity, \( R_c \) is associated to a purely imaginary coefficient \( c \), and then overstability appears.
Figures 3 and 4 represent collective critical Rayleigh curves for many values of \( \varphi \). In Figure 3 the Darcy’s number is 0.01 (as in Figures 2a - 2c), while in Figure 4 the Darcy’s number is 0.1. Qualitatively the plots are different, but the main differences are the presence of more stability islands and a richer interlacing of critical curves in which instability arises as exchange of stability or as overstability.

Since a Squire-like theorem has not been proven for this type of problems. We represent in Figures 5 and 6 the critical Rayleigh curves for general wave numbers for a fixed inclination of the plane chosen so that a stability island does exist. In this 3-dimensional plot the stability island is a tunnel ending in the \( b = 0 \) plane.

In Figure 7 we represent the threshold of nonlinear stability for the Lyapunov functional (24). Below is the horizontal plane \( R = (2\pi^2(1+Da\pi^2))(1+\cos\varphi) \), below which the basic solution is globally asymptotically stable.

In Figure 8 we finally plot the critical Rayleigh surface and the underlying threshold of nonlinear stability.

In Figure 9 we plot the critical linear Rayleigh number as a function of the inclination angle \( \varphi \) with a fixed Darcy’s number \( Da \). The various curves are associated to different choices of inertial coefficient \( G \). Observe that the critical curve associated to stationary convection is independent of \( G \) since the spectral parameter \( c \) multiplies the inertial term, hence when \( c = 0 \) such component plays no role. The critical curve associated to overstability is always increasing with the angle, but its dependence on \( G \) is not uniform. For angle above approximately \( 60^\circ \) the effects of \( G \) is not always stabilizing.

In conclusion, our analytical and numerical results show that:

- both the inclination and the Darcy’s numbers have stabilizing effects;
- the inertia term \( G \) does not modify the critical nonlinear Rayleigh numbers (the results we obtain are independent of \( G \));
- the critical Rayleigh numbers, in all the cases we have examined, are obtained for longitudinal or transverse perturbations;
- the critical values for \( Da = 0 \) are the same as those obtained by Rees and Bassom [1];
- the linear and nonlinear critical Rayleigh numbers are very close and, in the cases we have examined, they are obtained for streamwise perturbations;
- for linear instability associated to transverse perturbations, the disappearance of the stability island is associated to the jump discontinuities.
and to the transitions from stationary convection to overstability (see Figures 1, 9);

- for linear instability associated to transverse perturbations, increasing Da the transition from stationary convection to overstability take place at decreasing angles (see Figure 1);

- The influence of the inertial term $G$, in a limited number of cases (Da= 0.1), is stabilizing for small angles. For larger angles the influence of $G$ is more complex and requires further investigations.

- When Da = 0, the classical results of Rees [21] prove that transverse perturbations are always stable when the inclination angle is above approximately 31.5°. However, when Da > 0, above an angle depending on Da, instability sets in as oscillatory convection. This is a new mathematical and physical result.

Data accessibility. This work has no supplementary data.

Competing interests. We have no competing interests.

Authors contributions. P.F. carried out the linear instability computations and helped draft the concluding section; G.M. and A.G. contributed to the nonlinear stability section. All authors gave final approval for publication.

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Ethics. This work does not pose ethical issues.

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Figure 1: Transverse (spanwise) and longitudinal (streamwise) critical Rayleigh numbers for different inclination angles $\varphi$ and Darcy’s numbers $Da$. 
Figure 2a: Critical curves in the \((R, a)\) plane \((b = 0)\) for \(\bar{u}_1 = \bar{u}_2 = \sigma = 0\), \(Da= 0.01\), and different inclination angles. Thin and thick lines indicate transition to instability respectively via stationary convection or overstability.
Figure 2b: Same parameters and symbols of Fig. (2a).

Figure 2c: Same parameters and symbols of Fig. (2a).
Figure 3: Critical curves associated to transverse perturbations. Same parameters and symbols of the previous figures, and many inclination angles from $\varphi = 18.25^\circ$ up to $\varphi = 25.75^\circ$ with steps of $0.5^\circ$. The subfigures 2a-2c are particular curves in this collective plot.
Figure 4: Critical curves for transverse perturbations. Same parameters and symbols of previous figures, except for Darcy’s number $Da = 0.1$. Inclination angles varying from $\varphi = 18^\circ$ up to $\varphi = 26^\circ$ with steps of $0.5^\circ$. 
Figure 5: Critical three-dimensional surface for linear instability with respect to wave numbers $a$ and $b$ running from 0 to 4 and $R$ up to 500. Darcy’s number is chosen $Da = 0.01$ and the inclination angle is $\varphi = 24^\circ$. The instability region is the connected component laying above the surface.
Figure 6: Same parameters of previous figure, except the inclination angle $\varphi = 25.3^\circ$. This value is close to $\varphi = 25.5^\circ$, where the island of instability disapppears.
Figure 7: Comparison of the critical three-dimensional surfaces for nonlinear stability with Lyapunov functional (24) (above) and nonlinear stability estimate (22) (below) with respect to wave numbers $a$ and $b$ running from 0 to 4. Darcy’s number $Da = 0.01$ and inclination angle $\varphi = 24^\circ$. 
Figure 8: Comparison of the critical three-dimensional surfaces for linear instability (above) and nonlinear stability (below) with respect to wave numbers $a$ and $b$ running from 0 to 4. Darcy’s number $D_a = 0.01$ and inclination angle $\varphi = 24^\circ$. 
Figure 9: Transverse critical linear Rayleigh numbers as function of the inclination angle $\varphi$, for given Darcy’s number $Da = 0.01$, and different values of the inertial coefficient $G$ from $0$ to $10^{-3}$ with steps of $10^{-4}$. Thick lines indicate overstability, while the thin line indicate stationary convection. The dashed line is a visual guide associated to the transition from stationary convection to overstability.